DYNAMICS OF THE PIANOFORTE STRING AND THE HAMMER

PART V (SOME SPECIAL THEORIES)

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ABSTRACT. The general theory of the finite Pianoforte string struck by an elastic hammer has been already developed in different papers. In this paper, the approximate conditions, necessary to reduce the general theory to obtain special theories given by different workers previously, are discussed. Kaufmann has considered two cases in which the hard hammer strikes (*i*) very near the end of the string, and (*ii*) at the mid-point of the string. The case of the hard hammer is obtained by considering in general theory the elastic constant of the hammer as large as infinite. All the expressions necessary for case (*i*) are obtained by considering the longer segment as large as infinite, and case (*ii*) is obtained by making two segments of the string equal. The expressions obtained by Helmholtz are obtained by considering the tension of the string large enough compared to the elastic constant of the hammer. The expressions obtained by Bhargava-Ghosh are obtained by the same approximation in the case of a semi-infinite string. The expressions as given by Das's theory are obtained by considering the longer segment to be as large as infinity, and that of Delemer is obtained by putting infinite for the elastic constant of the hammer, and, the hammer is considered to be a massive one. In order to explain the effect of the velocity of impact on duration of contact, the impact is considered in the light of Hertz's theory. A good agreement between the theory and the experiment is obtained. In this case the pressure exerted by the hammer on the string becomes appreciable after a certain instant when it comes in contact and leaves the string long after the pressure falls to a very small value. Similar was the assumption made by Lamb in his theory.

INTRODUCTION

The dynamics of the pianoforte string and a hammer has been completely developed in different papers.[1] The theory has been tested by the experimental datas supplied by different authors. In this paper it will be shown that the older theories which fail to explain fully the different experimental facts will come out as special cases of the present general theory. The approximations necessary for simplification of the general theory to the respective special theories will show their range of applicability.

The symbols used in this paper are same as before.[1]

The expressions for the displacement of the struck-point, the pressure exerted and the corresponding duration of contact for an elastic hammer striking very near the end of a string are obtained in Part II [*vide* eq. (20)] and Part IV [*vide* eq. (27)].

$$y_a = \frac{Ev_0c}{T}\left[A^2e^{\alpha t} + \frac{A}{\nu}e^{\mu t}\sin(\nu t - \epsilon)\right] \qquad \ldots \quad (23)$$

$$-P = \frac{Am_1Ev_0c}{T\nu}\left[\nu A(\mu^2+\nu^2-2\mu\alpha)e^{\alpha t} - (\mu^2+\nu^2)e^{\mu t}\sin\left(\nu t - \epsilon + \tan^{-1}\frac{2\mu\nu}{\mu^2-\nu^2}\right)\right]$$

$$\Phi = \frac{1}{\nu}\left[\tan^{-1}\frac{\nu}{\mu-\alpha} - \tan^{-1}\frac{2\mu\nu}{\mu^2-\nu^2}\right] \qquad \ldots \quad (17)$$

where, $A = \frac{1}{\sqrt{(\alpha-\mu)^2+\nu^2}}$; $\quad \alpha = \frac{E}{2m}\cdot\frac{1}{\mu} + \frac{c}{\alpha}$; $\quad \mu = -\frac{T}{2m_1c}$;

$$\nu = \sqrt{-\frac{2\mu c}{a} - \mu^2}; \quad m_1 = m_0 + \frac{Tm}{Ea}; \quad m_0 = m + \frac{\rho a}{3};$$

$$\epsilon = \tan^{-1}\frac{\nu}{\mu-\alpha}.$$

The eq. (23) can easily be reduced to what is obtained by Bhargava-Ghosh[2] by making α infinite, *i.e.*, when magnitude of T is infinitely large compared to the magnitude of E. This makes the pressure of impact, at $t=0$ finite, which however cannot be true for elastic hammer. This assumption, which is necessary for the above reduction, is equivalent to, neglecting the term of third differential D^3y_a in the equation of motion of the struck-point, [*vide* eq. (24), Part II], which however was neglected by the authors in building up their theory cited above. They have further *assumed* that the shorter segment vibrates like a *rigid rod* during impact.

Das[3] has also got an expression similar to *eq.* (23), on Kaufmann's assumption for the 'mass correction', *i.e.*, the shorter segment behaves like a *rigid rod* during impact. But the present theory is free from any such assumption. It has been already pointed out (*vide* page 440, Part II), previously, that Das's deduction is also not free from criticism.

For a hard hammer $E \to \infty$. This reduces

$$\alpha \to \infty; \quad EA \to 1; \quad \epsilon \to 0;$$

$$m_1 \to m_0; \quad \mu \to -\frac{T}{2m_0c}; \quad \nu \to \sqrt{\frac{T}{m_0a} - \frac{T^2}{4m_0^2c^2}}.$$

As for the hard hammer $E=\infty$, the expressions for y_a, P and the duration of contact Φ becomes identical with those obtained by Kaufmann [*vide* eq. (7) and (9), and eq. (67), Part IV].

The general expression for the displacement of the struck point as obtained in Part III, eq. (52) is

$$
\begin{aligned}
y_a = f_1(t) &+ \sum_1^n [f_{n+1}(t-n\theta_1) - f_n(t-n\theta_1)] \\
&+ \sum_1^n [f_{n+1} - f_n](t-n\theta_2) \\
&+ 2[f_3 - 2f_2 + f_1](t-\Theta) \\
&+ \sum_1^n [(n+2)f_{n+3} - (3n+4)f_{n+2} + (3n+2)f_{n+1} - nf_n](t-\Theta-n\theta_1) \\
&+ \sum_1^n [(n+2)f_{n+3} - (3n+4)f_{n+2} + (3n+2)f_{n+1} - nf_n](t-\Theta-n\theta_2) \\
&+ 2[3f_5 - 9f_4 + 10f_3 - 5f_2 + f_1](t-2\Theta), \qquad \ldots (52)
\end{aligned}
$$

and the corresponding values of the pressure exerted by the hammer is [*vide* eq. (56), Part III]

$$
\begin{aligned}
-P = \frac{2T}{c} [f'_1(t) &+ \sum_1^n f'_{n+1}(t-n\theta_1) + \sum_1^n f'_{n+1}(t-n\theta_2) \\
&+ 2\{f'_3 - f'_2\}(t-\Theta) \\
&+ \sum_1^n \{(n+2)f'_{n+3} - 2(n+1)f'_{n+2} + nf'_{n+1}\}(t-\Theta-n\theta_1) \\
&+ \sum_1^n \{(n+2)f'_{n+3} - 2(n+1)f'_{n+2} + nf'_{n+1}\}(t-\Theta-n\theta_2) \\
&+ 2\{3f'_5 - 6f'_4 + 4f'_3 - f'_2\}(t-2\Theta)], \qquad \ldots (56)
\end{aligned}
$$

where each functions which represents disturbances produced during impact can be expressed in three different forms depending upon the nature of the values of q and p,

$$[q,\ p] = \frac{Ec}{4T} \mp \frac{1}{2} \sqrt{\left(\frac{Ec}{2T}\right)^2 - \frac{4E}{m}},$$

that is, according as

$$\left(\frac{Ec}{2T}\right)^2 > = < \frac{4E}{m}.$$

(*i*) For $\left(\frac{Ec}{2T}\right)^2 > \frac{4E}{m}$ we get [*vide* eq. (30), Part II]

$$f_1(t) = Av_0 \left[\frac{1}{q}(1-e^{-qt}) - \frac{1}{p}(1-e^{-pt})\right], \qquad \ldots (30)$$

$$f_2(t_1)=A^2 v_0\left[\frac{e^{-qt_1}}{q}(1-A+qt_1)+\frac{e^{-pt_1}}{p}(1+A+pt_1)\right], \quad \ldots \quad (30.1)$$

$$f_3(t_2)=A^3 v_0\left[\frac{e^{-qt_2}}{q}\left\{\tfrac{3}{2}(A-A^2)+\tfrac{1}{2}(3A-1)qt_2-\frac{q^2t_2^2}{2!}\right\}\right.$$

$$\left.+\frac{e^{-pt_2}}{p}\left\{\tfrac{3}{2}(A+A^2)+\tfrac{1}{2}(3A+1)pt_2+\frac{p^2t_2^2}{2!}\right\}\right], \ldots \quad (30.2)$$

etc., where $A=\dfrac{q+p}{q-p}$.

(*ii*) For $\left(\dfrac{Ec}{2T}\right)^2=\dfrac{4E}{m}$;

$$f_1(t)=\frac{2v_0}{q}\left[1-e^{-qt}(1+qt)\right], \quad \ldots \quad (36.1)$$

$$f_2(t_1)=\frac{4v_0}{q}\, e^{-qt_1}\frac{(qt_1)^3}{3!}, \quad \ldots \quad (36.2)$$

$$f_3(t_2)=\frac{8v_0}{q}\, e^{-qt_2}\left\{\frac{(qt_2)^4}{4!}-\frac{(qt_2)^5}{5!}\right\}, \quad \ldots \quad (36.3)$$

etc.

(*iii*) And for $\left(\dfrac{Ec}{2T}\right)^2<\dfrac{4E}{m}$

$$f_1(t)=\frac{2\mu}{\nu}.\, v_0\left[\frac{\nu}{\mu^2+\nu^2}-\frac{1}{\sqrt{\mu^2+\nu^2}}\, e^{-\mu t}\sin\left(\nu t-\tan^{-1}\frac{\nu}{\mu}\right)\right], \quad \ldots \quad (39.1)$$

$$f_2(t_1)=\frac{2\mu^2}{\nu^3}.\, v_0 e^{-\mu t_1}\left[\sin\nu t_1-\nu t_1\cos\nu t_1\right], \quad \ldots \quad (39.2)$$

$$f_3(t_2)=\frac{\mu^3}{\nu^5}.\, v_0 e^{-\mu t_2}\Big[\sqrt{(\mu^2+\nu^2)}\,\Big\{\nu^2 t_2^2\sin\left(\nu t_2-\tan^{-1}\frac{\nu}{\mu}\right)$$

$$+\nu t_2\cos\left(\nu t_2-\tan^{-1}\frac{\nu}{\mu}\right)\Big\}+\mu\{2\nu t_2\cos\nu t_2-3\sin\nu t_2\}\Big], \quad \ldots \quad (39.3)$$

etc.,

where $\mu=\dfrac{Ec}{4T}$; $\nu=\dfrac{1}{2}\sqrt{\dfrac{4E}{m}-\left(\dfrac{Ec}{2T}\right)^2}$.

Das [3] considered the case of a semi-infinite string and he admitted that the method adopted by him failed to consider the case of a finite string. Further, even for a semi-infinite string he was not able to consider the case when $\left(\frac{Ec}{2T}\right)^2 = \frac{4E}{m}$ as in the present theory.

For semi-infinite string $b \to \infty$ $l \to \infty$ and every function in eq. (52) and (56) whose argument contains $\theta_2\left(=\frac{2b}{c}\right)$, or $\Theta\left(=\frac{2l}{c}=\frac{2a}{c}+\frac{2b}{c}\right)$ must vanish, as these functions will not appear before the time $t=\frac{2b}{c}$, *i.e.*, if the hammer is assumed to leave the string before $t=\frac{2b}{c}$. This however is not the case for a massive hammer. Thus for a semi-infinite string

$$y_n = f_1(t) + \sum_1^n \left[f_{n+1}(t-n\theta_1) - f_n(t-n\theta_1)\right],$$

and

$$-P = f'_1(t) + \sum_1^n f'_{n+1}(t-n\theta_1). \qquad \dots \quad (83)$$

These are exactly the expressions obtained by Das.[3]

In the case of the soft hammer when $\left(\frac{Ec}{2T}\right)^2 < \frac{4E}{m}$ we have, by the help of eqs. (30), (39.1) and (56),

$$-P = \frac{4\rho v_0 c\mu}{\nu} e^{-\mu t} \sin \nu t \text{ during } 0 < t < \frac{2a}{c}, \qquad \dots \quad (84)$$

where μ and ν are given by eq. (38). When the hammer is very soft and light $\frac{Ec}{2T} \to 0$ so $\mu \to 0$ and $\nu \to \frac{E}{m}$ the eq. (84) reduces to

$$-P = \frac{Ev_0}{c} \sin \nu_0 t \text{ where } \nu_0 = \sqrt{\frac{E}{m}}. \qquad (85)$$

This is the form of the pressure function which Helmholtz[4] *assumed* in solving the problem of the Pianoforte string. In this connection *he remarked that the magnitude of* ν_0 *increases as the elastic power of the hammer increases and the weight decreases.* This, however, is evident from the above deduction.

In the case of the hard hammer, *i e.*, when $E \to \infty$ $q \to \frac{2\rho c}{m}$ and $p \to \infty$, we have for the interval $0 < t < \frac{2a}{c}$, from eqs. (30), (52), and (56),

$$P_1 = 2\rho v_0 c e^{-\frac{2\rho}{m}ct} \qquad \dots \quad (86)$$

$$y_a = \frac{mv_0c}{2T}\left(1-e^{-\frac{2\rho}{m}ct}\right) \qquad \dots \quad (86{\cdot}1)$$

$$y_1 = \frac{mv_0c}{2T}\left\{1-e^{-\frac{2\rho}{m}(ct+x-a)}\right\} \qquad \dots \quad (86.2)$$

$$y_2 = \frac{mv_0c}{2T}\left\{1-e^{\frac{2\rho}{m}(ct-x+a)}\right\} \qquad \dots \quad (86.3)$$

as $c^2\rho = T$.

We shall get all the expressions obtained by Delemer[5] *from the above eqs. (86), when the hammer is considered to be a massive one* so that $\frac{\rho}{m} \to 0$. By retaining the first two terms of the expansion $e^{-\frac{2\rho}{m}ct}$ in the above eqs. (86), we get

$$P_1 = 2\rho v_0 c \qquad \text{(a constant)}, \qquad \dots \quad (87)$$

$$y_a = \frac{P_1}{2T}\,ct, \qquad \dots \quad (87.1)$$

$$y_1 = \frac{P_1}{2T}\,(ct+x-a), \qquad \dots \quad (87.2)$$

$$y_2 = \frac{P_1}{2T}\,(ct-x+a). \qquad \dots \quad (87.3)$$

These equations are *same as given by Delemer who assumed in his theory that the pressure exerted by the hammer during impact is constant.* This, however, is the case of a massive hammer.

When a hard hammer ($E=\infty$) strikes at the mid-point of a string ($a=b$) and ($l=2a$) such that the pressure terminates during the second epoch, *i.e.*, $n=1$, we get, by putting $\theta_1=\theta_2$ in eqs. (32) and (56),

$$y_a = f_1(t) + 2[f_2(t+\theta_1) - f_1(t-\theta_1)]$$

$$= \frac{mv_0c}{2T}\left[\left(1-e^{-\frac{2\rho}{m}ct}\right) + 2\left\{1-e^{-\frac{2\rho}{m}(ct-2a)}\left\{1+\frac{2\rho}{m}(ct-2a)\right\}\right\}\right], \qquad \dots \quad (88)$$

and

$$P = \frac{2T}{c}\left[f'_1(t) + 2f'_2(t-\theta_1) - 2f'_1(t-\theta_1)\right]$$

$$= 2\rho v_0 c\left[e^{-\frac{2\rho}{m}ct} + 2e^{-\frac{2\rho}{m}(ct-2a)}\left\{1-\frac{2\rho}{m}(ct-2a)\right\}\right]. \qquad \dots \quad (89)$$

These are identical with those obtained by Kaufmann.[4]

It may be pointed out here that the theory given by Das does not take into account the reflection from the remoter end of the string. Das considered that the effect produced by the hammer when it strikes a finite string at a point dividing it into two unequal segments, was equal to the sum of two partial effects produced by the hammer, on two semi-infinite strings by striking at finite distances, which were equal to two segments respectively. This idea of Das leads to results different from[8] that of the Kaufmann for two equal segments, *i.e.*, when the hammer strikes at the mid-point. The difference arises after the time equal to the period of the vibration of the string measured from the beginning of the impact. This is due to the fact that Das has completely *ignored the effect of successive reflections of the waves from one end which has already suffered reflection from the opposite end.* It should, however, be noted that the present theory leads to the result identical with those of Kaufmann for all time.

The variation of the duration of contact with striking velocity in the case of the felt hammer was noticed by Weak[8] and Kaufmann and afterwards systematically studied by M. Ghosh.[9] In order to explain the above phenomenon we consider the period of impact to be divided into three *distinct* periods as Andrews[10] did in solving the collision problem of soft and elastic balls. It is assumed that in the first period the pressure exerted by the hammer-felt obeys Hertz's law and the string is not appreciably disturbed. After the compression has reached a certain value developing a finite pressure P_0, Hertz's law ceases to hold, and the string begins to be displaced. During this second period, the pressure exerted obeys Hooke's law, and the motion is given by the eqs. (19) and (19.1) and the corresponding dynamical behaviour is studied[1] previously. After the second period which may be called the 'Hooke's period' the extra compression, and so the corresponding pressure developed, is completely released, and the third period begins. As in this period Hertz's law is valid, it may be called the third 'Hertz-period.' It is assumed that the first and the third Hertz's periods have the same duration τ (say). Therefore the total duration of contact must be $\Phi_0 + 2\tau$ where Φ_0 is the duration of the second Hooke's period as calculated from our general theory in part IV, depending upon the *mass-ratios*, the striking length, and the elastic constant of the hammer, etc. Now the magnitude of τ is to be calculated.

The equation of motion of the hammer during the first or third 'Hertz-period' is

$$m\ddot{z} = m\ddot{y}_a + m\ddot{u} = -\epsilon u^{\frac{3}{2}}, \qquad \dots \quad (90)$$

where ϵ is a constant and $z = y_a + u$.

In passing we may remark E, that the elasticity of the hammer, which is taken to remain constant during the second 'Hooke's period,' is proportional to $u^{\frac{1}{2}}$ in 'Hertz-period' [compare eq. (19) and eq. (90)].

As the string is not appreciably disturbed in a 'Hertz-period,' we assume $y_a = 0$ approximately, so the eq. (90) becomes

$$\ddot{u} + \frac{\epsilon}{m} u^{\frac{3}{2}} = 0.$$

On integrating the eq. (91), and evaluating the constant from the condition that at $t=0$, $\dot{u}=v_0$ and supposing that the limiting values of u and $\dot{u}$ at the end of first 'Hertz-period' are u_0 and v respectively, we have

$$u_0 = \left(\frac{v_0^2 - v^2}{n}\right)^{\frac{2}{5}}, \qquad \ldots \quad (92)$$

where

$$n = \frac{4\epsilon}{5m}.$$

From eq. (92) we have for the time τ taken, to produce the compression u_0,

$$\tau = \frac{1}{v_0}\int_0^{u_0} \frac{du}{\sqrt{\left(1 - \frac{n}{v_0^2} u^{\frac{5}{2}}\right)}} = \frac{1}{n^{\frac{2}{5}} v_0^{\frac{1}{5}}} \int_0^{x_0} \frac{dx}{\sqrt{(1 - x^{\frac{5}{2}})}} \qquad \ldots \quad (93)$$

where

$$x_0 = n^{\frac{2}{5}} \cdot \frac{u_0}{v_0^{\frac{4}{5}}}. \qquad \ldots \quad (93.1)$$

After integrating term by term, we get

$$\tau = \frac{u_0}{v_0}\left[1 + \frac{n u_0^{\frac{5}{2}}}{7 v_0^2} + \frac{n^2 u_0^5}{16 v_0^4} + \frac{5 n^3 u_0^{\frac{15}{2}}}{136 v_0^6} + \frac{35}{704}\,\frac{n^4 u_0^{10}}{v_0^8} + + \right]. \qquad \ldots \quad (94)$$

As the total duration of contact Φ is given by

$$\Phi = \Phi_0 + 2\tau, \qquad \ldots \quad (95)$$

so

$$\Phi - \Phi_0 = \frac{2u_0}{v_0}\left[1 + \frac{n u_0^{\frac{5}{2}}}{7 v_0^2} + + \right], \qquad \ldots \quad (96)$$

where Φ_0 is calculated in the usual way from the pressure function, as given in part IV. As pointed out before, the algebraic solution of the pressure function for any value of E is rather difficult, so to verify the effect of the velocity of impact on the duration of contact we calculate Φ_0 taking $E = \infty$, *i.e.*, for a hard hammer. This value of Φ_0 will not introduce any serious error in showing the variation with velocity.

The data supplied by M. Ghosh[9] in this connection are used here to test the above theory giving the variation of Φ with velocity of impact.

The hammer of mass 21·2 gm. strikes at the mid-point of the string of length 600 cms. of line density 0·05 gm./cm. stretched under tension 38·5 kgms.wt.

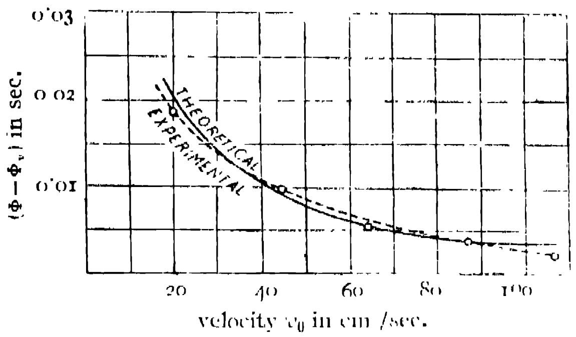

Figure 1.

The theoretical value of Φ_0 calculated from eq. (73) part IV is $2·09 \times 10^{-2}$ second, whereas the observed value is 3.080×10^{-2} seconds. The variation of $\Phi-\Phi_0$ with v_0 is calculated from eq. (96), taking only the first approximation and $u_0=0·2$ cm. Both experimental and theoretical variations are shown graphically. It is evident from the figure that the theory put forward agrees fairly well with the experiment.

It may be remarked here that the existence of the small pressure during Hertz periods at the beginning and at the end of the impact which is the basis of the above calculation naturally reminds one of *Lamb's assumption* that "*the pressure exerted by the hammer on the string becomes appreciable after a certain instant when it comes in contact and leaves the string long after the pressure falls to a very small value.*"

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